CE 297: Problems in the Mathematical Theory of Elasticity: Homework V

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An (*) indicates somewhat harder problems.

1. Use the Schwarz reflection principle and the Plemelj-Sokhotski formulae to re-derive the necessary and sufficient conditions for a continuous function $\omega(t)$, defined on a circle $|t| = R$, to be the boundary value of a function

   (i) holomorphic in S^+ and

   (ii) holomorphic in $S^- \cup \mathbb{C}_\infty$

   where S^+ and S^- are, respectively, the interior and exterior of the circle.

2. Evaluate the Cauchy integral
$$\frac{1}{2\pi i}\oint_{\mathcal{L}} \frac{\bar{t}^{\,n}}{(t-z)}\,dt$$
   where $\mathcal{L}$ is the circle $|t| = R$, and $n \geq 0$ is an integer, for

   (i) $z \in S^+$ (ii) $z \in S^-$

3. Using the Cauchy integral operator, formulate the traction (first) boundary value problem for a circular disk using the functions $\Phi(z)$ and $\Psi(z)$ rather than $\varphi(z)$ and $\psi(z)$, as done in class.

4. Show that the condition that the constant a_1 (the coefficient of z in the series expansion of $\varphi(z)$ for a disk) be purely real for the traction boundary problem is equivalent to the condition that the resultant moment due to the applied tractions vanishes.

5. For the problem of the disk with equal and opposite forces applied at the ends of a chord, evaluate the stresses σ_{xx}, σ_{yy}, and σ_{xy}. Do not use regular polar coordinates[1];

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[1]This leads to extremely complicated expressions

instead introduce a double-polar coordinate system r_1, θ_1 and r_2, θ_2 centered at the points of load application z_1 (B) and z_2 (C), i.e. $z_1 - z = r_1 \exp(-i\theta_1)$, $z_2 - z = -r_2 \exp(i\theta_2)$. Note that using this definition the angles θ_1 and θ_2 are intended to be measured clockwise and counterclockwise, respectively, from the chord BC.

6. *Find the potentials φ and ψ for the problem of a circular disk loaded by two *internal*, horizontal point loads. Assume that one load, $(F, 0)$ acts at (0,0) and another load $(-F, 0)$ acts at $(a, 0)$, where $a < R$. To solve this problem, you will need to write down potentials corresponding to these two point loads, and then find additional potentials that clear the tractions that they cause along the boundary of the disk.

7. In the problem of a partially pressurized hole discussed in class, derive expressions for both $\phi(z)$ and $\psi(z)$. Show all steps.

8. Using the Cauchy integral operator, formulate the traction (first) boundary value problem for an infinite plate with a circular hole using the functions $\Phi(z)$ and $\Psi(z)$ rather than $\varphi(z)$ and $\psi(z)$. You might find it helpful to write down the applied traction b.c. using the form $N(\theta) - iT(\theta)$ as before.

9. A rigid disk of radius R is welded with the edge of a hole of the same radius in an infinite plate. A pure shear stress $\sigma_{xy}^{\infty} = T$ is applied to the plate at infinity. Find the stresses in the plate.

10. A rigid disk of radius $R + \Delta$, where $0 < \Delta << R$ is interference fitted into an elastic plate with a hole of radius R. Find the stresses and displacements in the plate.

11. A rigid disk of radius R is welded along the boundary of an infinite plate with a circular hole, also of radius R. The inclusion is given a small horizontal displacement $\Delta > 0$. Find the stresses in the plate.

    Hint: Since the problem is ill-posed without specifying a resultant force, assume an unknown horizontal force $+F$ and proceed. You should find that $F \propto \Delta$.

12. Use the solution of the point force applied to a welded rigid inclusion in the limit when the hole / inclusion radius R goes to 0. Do you recover Kelvin's solution for a point load applied at the origin ($z_0 = 0$)?

13. *An elastic disk of radius a is compressed radially and inserted into a hole of slightly smaller radius R ($R < a$) in an infinite elastic plate. The disk has elastic moduli μ_I, κ_I and the plate μ_{II}, κ_{II}. Find the pressure transmitted across the interface.

    Examine and discuss the following limiting special cases: (i) A nearly rigid disk (ii) A nearly rigid plate (iii) Elastically similar disk and plate

    Hint: Note that there are two different elastic bodies in this problem.